One Round Voronoi Game on Grid

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Abstract

In this paper, we study the discrete Voronoi game on a rectangular grid $G(m,n)$ where two players place their facilities (k facilities each) in the grid in *one round*. The area of the grid is shared among the two players (White and Black, denoted by W and B, respectively) based on the nearest neighbor rule with the *Manhattan* metric. We show that in *one dimensional* grid, $G(1,n)$, W as the first player, has a winning strategy which will guarantee the winning margin of one for him if $2k$ does not divide n. Then, we study *two dimensional* grid, $G(m,n)$, where $m > 1$. In this case B is able to defeat W in some cases. Therefore we calculate the properties of a grid in which W wins the game when m is an odd number. Furthermore, we propose a lower bound for the grid size where W wins the game with winning margin of at least m. When m is even, W is not able to win the game assuming the optimal play by B (the best case for W is to not lose).

1 Introduction

Facility location is an optimization problem, dealing with placing a set of facilities which must serve a set of customers based on an optimality measure. Adding the *competitive market players* to this context and combining it with the arguments of *game theory* will lead to *competitive facility location* problem. This problem has been extensively studied in different fields such as computational geometry, mathematics, industrial engineering and

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operation research. Extensive overviews of this problem and its models can be found in the surveys by Friesz et al. [8] and Eiselt and Laporte [7]. The *Voronoi game* is a simple geometric model for the competitive facility location problem. Two competing market players in Voronoi game, place two disjoint set of facilities in the bounded gaming workspace in a given number of rounds. The Voronoi diagram of all sites is calculated, and at the end, the player whose total Voronoi region occupy the most area, is considered the winner.

From the viewpoint of the number of rounds, there are two types of Voronoi game. In the *one round* game, the first player (White denoted by W) places a set of k facilities. After W's play, the second player (Black denoted by B) places his set of k facilities in the game region. In the other variation which is called *k-round* game, the two players place one facility each alternately for k rounds in the game region. Voronoi game has been widely studied on the continuous space domain. One dimensional k-round Voronoi game where the game region is a line segment or a circle, was studied by Ahn et al. [1]. By following their proposed strategy, the second player (B) always wins the game by a winning margin of $\epsilon > 0$. Their defined k-round game is different from the *one* round game on the continues line segment where W can achieve a win by placing his facilities at the odd integer points. Also, like the k-round case, W can decrease the loss margin as much as he wishes. Fekete and Meijer [2] proposed a model for two dimensional one round game played on a rectangular continuous demand region. The solution for some of the open problems in the one round Voronoi game in two dimensional grid can be found in [2]. In particular, they proposed a characterization of the game with the assumption of optimal play. They also studied the winning conditions in terms of the number of facilities and the aspect ratio of the game board.

The discrete Voronoi game was introduced by Teramoto et al. [3]. In this game, two players place n facilities each, in a graph which contains at least $2n$ nodes. They showed that in a complete $k - ary$ tree, where the tree is large enough considering n and k, the first player has a winning strategy. The Voronoi game on graphs and particularly on trees were later studied by Kiyomi et al. [4]. They proposed a model for the different cases of the game on a path. They showed that the game on a path containing n vertices where the game play is continued for $t < \frac{n}{2}$ rounds will finish in an equal state for two players if either n is even or t is not one. When n is odd and $t = 1$, the first player wins the game. Banik et al. [5] studied another variation of the discrete Voronoi game which is played on a *simple polygon*. They proposed the complexity results for the both players when the

number of facilities for each player is limited to one. They also studied the problem of one round discrete game on a line segment [6]. In this problem, the players are competing for owning a set of n users by placing a set of m points each and disjoint from each other, on the line segment in one round. They proved that if the sorted order of the n points on the line segment is known, the optimal strategy for the second player can be computed in $O(n)$ time. They also showed that the optimal strategy of the first player for any $m \geq 2$ can be computed in $O(n^{m-\lambda_m})$ time where $0 < \lambda_m < 1$ is a constant depending to m. This problem on the multi round version and also on the multi dimensional case is still open. The one round discrete Voronoi game in $\mathbb{R}^2$ in presence of existing facilities were investigated by Banik et al. [9]. They studied a simplified variant of the discrete Voronoi game in the plane. The game consists of two players and a finite set of users in the plane. Moreover, the two players have already placed a set of facilities F and S, respectively, in the plane. The game is started by the first player by placing a new facility followed by the second player, placing another facility. This is very similar to the final round of a k-round discrete Voronoi game when both players had placed a set containing $k-1$ facilities each, already. They proposed polynomial time algorithms for finding the optimal strategies for both of the players when the location of the points in F and S were selected arbitrarily. They showed that the optimal strategy for the second player and the first player, given any placement for the first player, can be computed in $O(n^2)$ and $O(n^8)$ time, respectively.
In this paper we study the one round discrete Voronoi game on a grid $G(m, n)$. To achieve a better model, the facilities are considered to have area. The problem is studied in one dimensional grid first and a winning strategy that guarantees the winning margin of one if $2k \nmid n$ is proposed for W. Further, the optimality of W's strategy is shown as well. Two dimensional case where the width of the grid, m, is an odd number is studied as well, and the condition for the W's win is computed. These computations provide conditions in a way that W wins the game by a margin of m at least if he follows the proposed strategy. It is clear that in the grid with even m, the symmetry play by B finishes the game in a tie in most cases. However, proposing a winning strategy in even m case for B, seems much harder.
The rest of this paper is organized as follows. In the next section, the game definitions and formulation are discussed. Section 3 covers one dimensional Voronoi game on the grid. The game in two dimensional grid board is discussed in section 4. Finally, the last section summarizes some open issues which are introduced by this problem.

2 Voronoi Game on Grid

We start the discussion by the formulation of the game. The *grid Voronoi game* is denoted by $GVG_r(G, k)$ in which, k is the number of facilities for either of the players and r is the number of play rounds. In the rest of this paper, $G(m, n)$ is considered the game play board. G is a rectangular *grid* with the length of n and the width of m and consists of $m \times n$ unit squares called *cell*s. Also, all of the distances in the game are measured using the *Manhattan* metric. These will change the calculations for the Voronoi diagram. In the one round game variation ($r = 1$) which is studied in this paper, each of the players (White denoted by W as the first player and Black denoted by B as the second player) chooses a set of k facilities disjoint from each other. Players are not able to choose a cell which is already occupied by either of the players. One or both of the players will own the whole area or a part of a cell respectively based on the nearest neighbor rule. Hence the area of a cell which has the same distance from some cells occupied by W or B, is shared among them. Furthermore, By placing a facility in a cell, the corresponding player will own all the area of that specific cell. The players compete to own the maximum part of the $m \times n$ cells of the game region in one round and the player owning the larger part of the region is the winner of the game.

3 One Dimensional Grid Voronoi Game

W has a winning strategy in the one-round Voronoi game on a continues line segment [1]. This player wins the game by placing his facilities at the odd integer points along the line segment. In this settings, B can decrease the loss margin as much as he wishes. In the discrete case a winning strategy for W and a defense strategy for B were proposed [6].

In this section, we consider $G(1, n)$ as a one dimensional grid with the length of n (and the width of $m = 1$). Without loss of generality suppose that the orientation of the grid is horizontal as illustrated in Figure 1. Therefore, we have the following definitions:

Definition 1. *The distance between the cells of two consecutive facilities of W is called an interval.*

Definition 2. *The horizontal distance between the left side of the game region and the leftmost occupied cell of W is called left half interval and is denoted by LHI. Likewise, the half interval between the right side of the*

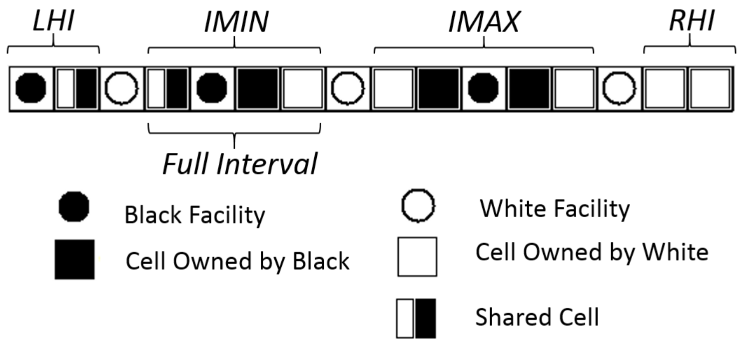


Figure 1: *1-D* grid Voronoi game $GVG_{r=1}(G(m=1, n=16), k=3)$

game region and the rightmost occupied cell of W is called right half interval and is denoted by RHI.

Notation 1. *The length of every full/half interval I is denoted by $|I|$.*

By considering these definitions, we show that selecting the position of facilities according to

$$\left\lfloor \frac{(2i-1) \times n}{2k} \right\rfloor \quad ; i = 1, ..., k \tag{1}$$

in $GVG_1(G(1,n),k)$ is a winning strategy for W. To prove this, we need the following propositions and definitions. Note that, they are true if W uses this placing strategy.

Proposition 1. *The maximum length of a full interval, in case of existence, is $\lfloor \frac{n}{k} \rfloor$.*

Proof. For any optional $1 \leq t \leq k-1$, the Manhattan distance between two consecutive occupied cells of W (t^{th} and $(t+1)^{th}$) is:

$$\begin{aligned} &\left\lfloor \frac{(2t+1) \times n}{2k} \right\rfloor - \left\lfloor \frac{(2t-1) \times n}{2k} \right\rfloor - 1 \\ &\leq \frac{(2t+1) \times n}{2k} - \left(\frac{(2t-1) \times n}{2k} - 1 \right) - 1 = \frac{n}{k}. \end{aligned}$$

It is obvious that the distance between any two optional cells is an integer number. Hence the maximum distance, in case of existence, is $\lfloor \frac{n}{k} \rfloor$. $\square$

In the following, an interval with the maximum length is denoted by IMAX.

Proposition 2. *The minimum length of a full interval, in case of existence, is* $\lfloor \frac{n}{k} \rfloor - 1$.

Proof. With the same reasoning as Proposition 1, the following equations are valid when we subtract the positions of two consecutive occupied cells:

$$\begin{aligned}&\left\lfloor \frac{(2t+1)\times n}{2k} \right\rfloor - \left\lfloor \frac{(2t-1)\times n}{2k} \right\rfloor - 1 \\ \geq &\left\lfloor \frac{(2t+1)\times n}{2k} - \frac{(2t-1)\times n}{2k} \right\rfloor - 1 = \left\lfloor \frac{n}{k} \right\rfloor - 1.\end{aligned}$$

As a result, the minimum interval length, in case of existence, is $\lfloor \frac{n}{k} \rfloor - 1$. □

Likewise, IMIN indicates a full interval with the minimum length.

Proposition 3. *For any optional* n *the inequality* $|RHI| \leq |LHI|$ *holds. As a result* $|RHI| + |LHI| \leq \lfloor \frac{n}{k} \rfloor$.

Proof. According to Eq. (1), counting the grid cells horizontally, is started from zero. Hence, the length of LHI is:

$$|\text{LHI}| = \left\lfloor \frac{n}{2k} \right\rfloor. \tag{2}$$

To measure the length of LHI, let $i = k$ in Eq. (1). The horizontal position of k^{th} occupied cell is: $\lfloor n - \frac{n}{2k} \rfloor$. This will result the following calculations:

$$n - \left\lfloor n - \frac{n}{2k} \right\rfloor - 1 = n - n - \left\lfloor -\frac{n}{2k} \right\rfloor - 1 = -\left\lfloor -\frac{n}{2k} \right\rfloor - 1.$$

Then,

$$-\frac{n}{2k} + 1 \geq -\left\lfloor \frac{n}{2k} \right\rfloor \Rightarrow \left\lfloor -\frac{n}{2k} \right\rfloor + 1 \geq -\left\lfloor \frac{n}{2k} \right\rfloor$$

$$\Rightarrow |\text{RHI}| = -\left\lfloor -\frac{n}{2k} \right\rfloor - 1 \leq \left\lfloor \frac{n}{2k} \right\rfloor = |\text{LHI}|.$$

Furthermore, according to $|\text{RHI}| \leq |\text{LHI}| = \lfloor \frac{n}{2k} \rfloor$, the following holds:

$$|\text{LHI}| + |\text{RHI}| \leq 2 \times \left\lfloor \frac{n}{2k} \right\rfloor \leq \left\lfloor \frac{n}{k} \right\rfloor$$

□

Proposition 4. *B owns at least $|LHI|$ of the game region by placing a facility in an IMIN interval. This means that selecting LHI or RHI is dominated by the selection of an empty IMIN interval.*

Proof. B gains

$$\frac{\left\lfloor \frac{n}{k} \right\rfloor - 2}{2} + 1 = \frac{\left\lfloor \frac{n}{k} \right\rfloor}{2} \tag{3}$$

by placing a facility in an IMIN interval. This will lead to the correctness of the proposition:

$$\left\lfloor \frac{n}{k} \right\rfloor - 1 \leq 2 \times \left\lfloor \frac{n}{2k} \right\rfloor \leq \left\lfloor \frac{n}{k} \right\rfloor \Rightarrow \left\lfloor \frac{n}{2k} \right\rfloor \leq \frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor .$$

□

Proposition 5. *Placing two facilities in one IMIN interval is not an efficient placing strategy for B.*

Proof. According to Proposition 2, the length of any IMIN interval is $\left\lfloor \frac{n}{k} \right\rfloor - 1$. B will own at most the entire length of the interval by placing two facilities in any interval. However, this selection is dominated by placing a facility in an IMIN interval and placing another one in LHI, because:

$$\left\lfloor \frac{n}{k} \right\rfloor - 1 \leq \left\lfloor \frac{n}{2k} \right\rfloor + \frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor \leq \left\lfloor \frac{n}{k} \right\rfloor .$$

Furthermore, the result of placing two facilities in two empty IMIN intervals is $\left\lfloor \frac{n}{k} \right\rfloor$. □

Proposition 6. *Placing two facilities in one IMAX interval is not efficient for B.*

Proof. According to Proposition 1, the length of any IMAX interval is $\left\lfloor \frac{n}{k} \right\rfloor$. B gains this amount by placing two facilities in two IMIN intervals if they exist. Placing a facility in an IMIN and the other one in an IMAX interval guarantee $\left\lfloor \frac{n}{k} \right\rfloor + \frac{1}{2}$ Voronoi region for B. Furthermore, playing a facility in IMAX and the other one in LHI is at least as efficient as placing two facilities in IMAX, because:

$$\frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n}{2k} \right\rfloor + \frac{1}{2} \geq \left\lfloor \frac{n}{k} \right\rfloor .$$

□

Theorem 1. *W wins $GVG_1(G,k)$ in $G(1,n)$ by selecting the position of his facilities according to Eq.* (1) *where $2k \nmid n$. The game ends in a tie when $2k \mid n$.*

Proof. Assume that t is the number of IMIN intervals when W places his facilities according to Eq. (1). The number of IMAX intervals will be $k-1-t$. Considering Propositions 1 to 6, B is forced to place a facility in each interval and finally places a facility in LHI. Hence, the Voronoi region of B is

$$t \times \left(\frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor\right) + (k-1-t)\left(\frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor + \frac{1}{2}\right) + \left\lfloor\frac{n}{2k}\right\rfloor.$$

This method of placement of facilities is called **simple strategy** from now on.
The Voronoi region of W has two cases:

- $2k \mid n$ **:** Then, $|\text{RHI}| < |\text{LHI}|$ and as a result,

$$t \times \left(\frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor - 1\right) + (k-1-t)\left(\frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor - \frac{1}{2}\right) + \left\lfloor\frac{n}{2k}\right\rfloor - 1 + k.$$

- $2k \nmid n$**:** This means that $|\text{RHI}| = |\text{LHI}|$ and W wins

$$t \times \left(\frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor - 1\right) + (k-1-t)\left(\frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor - \frac{1}{2}\right) + \left\lfloor\frac{n}{2k}\right\rfloor + k.$$

Subtracting the Voronoi region amount of W and B in the first case will finish the game in tie. W wins the game with the winning margin of one in the later. □

Theorem 1 fully covers $GVG_1(G,1)$ on $G(1,n)$. This theorem confirms the initial insight about the problem. Regular segmentation of G equally as much as possible, is a winning strategy for W.

3.1 Proof of Optimality

In this section we prove that the placing strategy based on Eq. (1) is an optimal placement strategy for W. It is clear that different arrangements of IMIN and IMAX intervals between LHI and RHI are also optimal placement strategies if placement based on Eq. (1) is optimal. When we have t objects of one kind (IMAX intervals) and $k-1-t$ objects of another kind (IMIN intervals), then the number of ways of arranging them in a row (number of optimal placement strategies) is equal to:

$$\frac{(k-1)!}{t!(k-1-t)!}.$$

Since different arrangements of IMIN and IMAX intervals are equivalent, we use Eq. (1) in the following.

Theorem 2. *Placing facilities according to Eq.* (1) *is an optimal placement strategy for W.*

Proof. Suppose that W uses an arbitrary placing strategy other than Eq. (1) (or any of its other variations) in $G(1,n)$. Also, we denote the length of the created half/full intervals by $L_0, L_1, ..., L_k$ from the left to the right side of the grid respectively assuming the grid is horizontal. It is clear that by placing a facility in each one of the W's intervals, the amount of Voronoi region of B is

$$\mathrm{MAX}(L_0, L_k) + \frac{L_1+1}{2} + \frac{L_2+1}{2} + ... + \frac{L_{k-1}+1}{2}.$$

Simplifying this equation leads to the following:

$$\mathrm{MAX}(L_0, L_k) + \frac{1}{2}\left(L_1 + L_2 + ... + L_{k-1}\right) + \frac{k-1}{2} \tag{4}$$

Similarly, the Voronoi region of W is equal to

$$\mathrm{MIN}(L_0, L_k) + \frac{1}{2}\left(L_1 + L_2 + ... + L_{k-1}\right) + \frac{k+1}{2} \tag{5}$$

the result of subtracting Eq. (4) from Eq. (5) is:

$$\mathrm{MIN}(L_0, L_k) - \mathrm{MAX}(L_0, L_k) + 1. \tag{6}$$

If $\mathrm{MIN}(L_0, L_k) \neq \mathrm{MAX}(L_0, L_k)$ holds, B does not lose the game (because $\mathrm{MIN}(L_0, L_k) \leq \mathrm{MAX}(L_0, L_k)$). As a result, since $|\mathrm{LHI}| = |\mathrm{RHI}|$ must holds, the loss margin of B is not more than one, if he plays optimally. Also, note in previous equations that the length of every interval is at least one. Otherwise, B always can achieve equality with symmetry play (since the number of intervals is less than $k+1$). Now, suppose that the length of one of the intervals I is bigger than $|\mathrm{IMAX}|$ ($|I| = |\mathrm{IMAX}| + L$). We investigate this problem in two cases:

- $L \geq 2$: First, suppose that $L = 2$. B gains

$$\frac{|\text{IMAX}| + 3}{2}$$

by placing one facility in this interval. The remaining length of this interval for W is

$$\frac{|\text{IMAX}| + 1}{2}.$$

Now, suppose that $L_0 = L_k < \left\lfloor \frac{n}{2k} \right\rfloor$. B gains at least equality by placing two facilities in I. If $L_0 = L_k > \left\lfloor \frac{n}{2k} \right\rfloor$, placing a facility in an IMIN interval (there exist at least one if $k > 2$) is not efficient, because

$$|\text{LHI}| = |\text{RHI}| \geq \left\lfloor \frac{n}{2k} \right\rfloor > \frac{1}{2} \left\lfloor \frac{n}{k} \right\rfloor.$$

Hence,

$$|\text{LHI}| = |\text{RHI}| = \left\lfloor \frac{n}{2k} \right\rfloor.$$

Since $\left\lfloor \frac{n}{2k} \right\rfloor < \frac{|\text{IMAX}|+1}{2}$, placing two facilities in I when $|I| \geq |\text{IMAX}|+2$ guarantees equality for B. The same reasonings are valid for $L > 2$ as well.

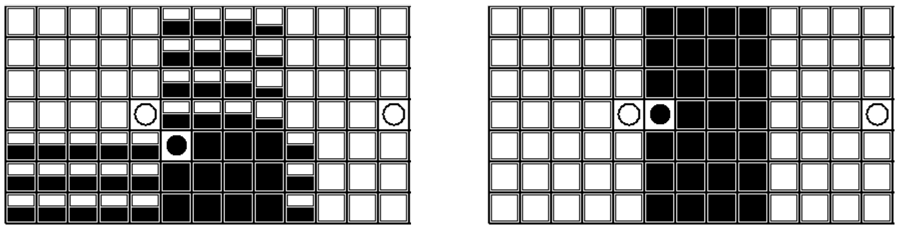
- $L = 1$: At first, assume that if W uses Eq. (1), the number of IMAX intervals is just one. Similar calculations enforces that $|\text{LHI}| = |\text{RHI}| = \left\lfloor \frac{n}{2k} \right\rfloor$. Therefore, if there exists an interval I with the length of $|\text{IMAX}| + 1$, there should exist at least one interval with the length smaller than $|\text{IMIN}|$ (say $|\text{IMIN}| - 1$). If this is the case, B again achieve at least equality by placing two facilities in I. If the number of IMAX intervals is more than one, existence of an interval with the length smaller than $|\text{IMIN}|$ is not guaranteed (W can create I with length $|\text{IMAX}| + 1$ by converting one of the IMAX intervals to IMIN). B gains $\frac{|\text{IMAX}|+1}{2}$ by placing a facility in I and $\frac{|\text{IMAX}|}{2}$ remains for W. Since $\frac{|\text{IMAX}|}{2} \geq \left\lfloor \frac{n}{2k} \right\rfloor$, B ends the game in an equal state when $\frac{|\text{IMAX}|}{2} > \left\lfloor \frac{n}{2k} \right\rfloor$ and W wins the game with margin of one when $\frac{|\text{IMAX}|}{2} = \left\lfloor \frac{n}{2k} \right\rfloor$ in this case ($GVG_1(G(1, 19), 4)$ as an example).

A similar proof is valid when there exist an interval I so that $|I| \leq |\text{IMIN}| - 1$. $\square$

Therefore, placing facilities based on Eq. (1) is an optimal placing strategy for W.
By proving theorem 2, we covered one round Voronoi game on grid completely. As a result, we proposed an optimal strategy for W in one round one dimensional Voronoi game having winning margin of at least one, if $2k \nmid n$. We also proved that W at least gain equality if $2k \mid n$. In the next section, we will cover two dimensional grid Voronoi game.

4 Two Dimensional Grid Voronoi Game

The game play scenario in two dimensional game is different. Both of the players can freely choose the location of their facilities in two directions and as a result the winning strategies will change. However, two dimensional grid Voronoi game is fundamentally different from two dimensional Voronoi game on the continuous region. Fekete and Meijer [2] proposed winning strategies and conditions for the one round game played on the continuous two dimensional region. By means of their proposed strategy and calculating the aspect ratio of the game region, one can predict the result of the game. Playing based on their strategy, will result in the winning margin of arbitrary small number $\epsilon > 0$. The winner of the game is decided by considering the aspect ratio of the game region and the number of facilities that each user can play.
This kind of reasonings is not valid in the grid. Actually, as the facilities in the grid Voronoi game have area, proposing winning strategy is much harder. The same limitations is the cause of non-symmetry in many grids. Therefore, by only considering the aspect ratio and the number of facilities for each user one can not determine the result of the game. Furthermore, in grid Voronoi game, more precise winning margin can be calculated and unlike the continuous case, none of the players can decrease the loss margin arbitrarily. Hence, we will divide the problem into two sub problems: the grid with odd width (m is odd) and the grids with even width (m is even). In the following, the winning condition for W will be proposed. Note however that, these conditions does not mean that B wins the game in the rest of cases (Unlike continuous region). For the grid with the even width, proposing a winning strategy for either of the players seems much more difficult. It is not hard to show that B does not lose the game in the grid with even width (symmetry play in many cases). Yet proposing a winning strategy for B when m is even is still open.



(a) Not in R_{mid} (b) In R_{mid}

Figure 2: B places a facility in a neighboring cell of the W's leftmost occupied cell in the leftmost full interval (odd length).

In this section suppose that $m \geq 3$ is an odd number. We denote the $\left(\frac{m+1}{2}\right)^{th}$ row of the grid (assuming the grid is horizontal) by R_{mid} and we call it the *middle row*. Furthermore, like one dimensional case, the horizontal distance between two consecutive facilities of W (which is a rectangle now), is called an interval (similar definitions for the half intervals also holds). In this section, we assume that W will place his facilities according to Eq. (1) horizontally and in R_{mid} vertically. Therefore, the position of every facility of W is selected based on the following equation:

$$\left(\frac{m+1}{2}, \left\lfloor \frac{(2i-1)\times n}{2k} \right\rfloor\right) \quad ; i = 1, ..., k. \tag{7}$$

Lemma 1. *Let $n_1 = \frac{5}{3}m \times k - \frac{7}{3}k + 1$ and W places his facilities in $G(m, n)$ according to Eq. (7). Also, suppose that B has placed a facility in R_{mid}, in a full interval, (Figure 2(b), Figure 3(b) and Figure 4). For every $n \geq n_1$, this place is the most efficient place for the B's facility in that interval.*

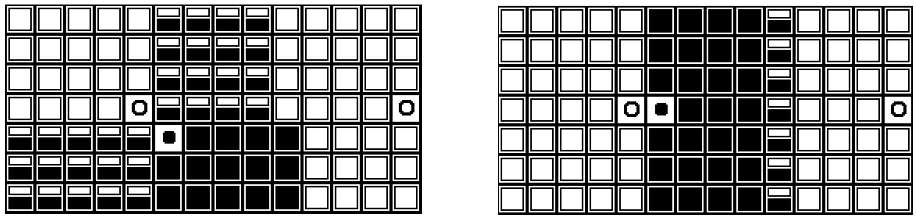
Proof. To prove this lemma, we investigate different possible cases. For the first case, assume that B places a facility in a neighboring cell of the W's leftmost occupied cell in the leftmost full interval with the odd length (Figure 2).

In this figures we denote the full interval by I and the half interval by HI. Suppose that $|I|$ is an odd number. Then the Voronoi region of B's facility in the left side of Figure 2 is:

$$\left(\frac{m-1}{2}\right) \times \left(\frac{|\mathrm{HI}|+1}{2} + \frac{|I|+2}{2}\right) + \left(\frac{m+1}{2}\right) \times \left(\frac{|I|-1}{4} + \frac{1}{3}\right). \tag{8}$$

Simplifying Eq. (8) results the following:

$$\left(\frac{3m-1}{8}\right) \times |I| + (m-1) \times \frac{|\mathrm{HI}|}{4} + \frac{19}{24}m - \frac{17}{24}. \tag{9}$$



(a) Not in R_{mid} (b) In R_{mid}

Figure 3: B place a facility in a neighboring cell of the W's leftmost occupied cell in the leftmost full interval (even length).

The Voronoi region for B's facility in the right side of Figure 2 is:

$$m \times \left(\frac{|I|+1}{2} \right). \tag{10}$$

Subtracting these two equations, considering $|\mathrm{HI}| = \frac{|I|-1}{2}$ or $|\mathrm{HI}| = \frac{|I|+1}{2}$ will result Eq. (11)

$$\begin{cases} n \geq \frac{2}{3}m \times k - \frac{4}{3}k & ; I = \mathrm{IMIN}\,, |\mathrm{HI}| = \frac{|I|-1}{2} \\ n \geq \frac{5}{3}m \times k - \frac{7}{3}k & ; I = \mathrm{IMIN}\,, |\mathrm{HI}| = \frac{|I|+1}{2} \\ n \geq \frac{2}{3}m \times k - \frac{7}{3}k & ; I = \mathrm{IMAX}\,, |\mathrm{HI}| = \frac{|I|-1}{2} \\ n \geq \frac{5}{3}m \times k - \frac{10}{3}k & ; I = \mathrm{IMAX}\,, |\mathrm{HI}| = \frac{|I|+1}{2} \end{cases} \tag{11}$$

Similar reasonings for the case demonstrated in Figure 3 in which $|I|$ is even leads to:

$$\begin{cases} n \geq m \times k - 2k & ; I = \mathrm{IMIN}\,, |\mathrm{HI}| = \frac{|I|}{2} \\ n \geq m \times k - 3k & ; I = \mathrm{IMAX}\,, |\mathrm{HI}| = \frac{|I|}{2} \end{cases} \tag{12}$$

Taking maximum over all the possible cases in Eq. (11) and Eq. (12), will result the proof of the lemma. Note that in the maximum case which the maximum value is

$$n_1 = \frac{5}{3}m \times k - \frac{7}{3}k + 1 \tag{13}$$

the corresponding interval, I, is an IMIN interval and $|I|$ is odd.

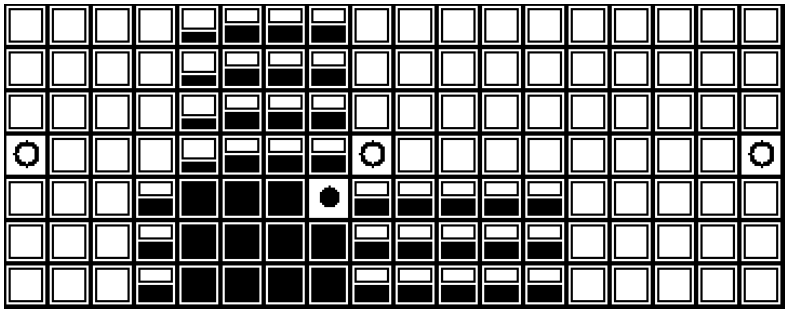


Figure 4: Placing B's facility beside W's facility when closest neighboring interval is a full interval

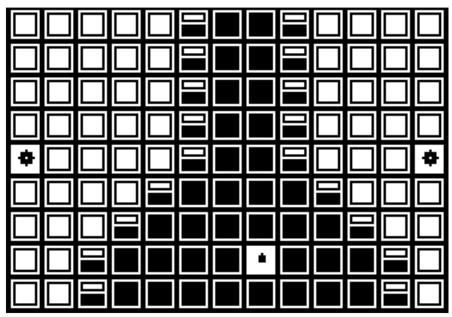


Figure 5: The whole Voronoi region of a facility is inside an interval

A similar proof for the case where B's facility is placed in a full interval and its closest neighboring interval is also a full interval (As illustrated in Figure 4) is similar and leads to the same n_1 as a result. □

In the following lemma, another case of inefficiency of not selecting R_{mid} for B's play is discussed.

Lemma 2. *Assume that B, places a facility in an interval I in a way that the whole Voronoi region of that facility remains inside the bounds of I. Also suppose that the vertical distance of this facility to R_{mid} is $a > 0$. Transferring this facility vertically to R_{mid} will increase the Voronoi region and the amount of increment is a^2.*

Proof. Suppose that B, places a facility in an interval I in a way that the whole Voronoi region of that facility remains inside the bounds of I (Figure 5). To measure the Voronoi region of this facility, we split the Voronoi region into three parts vertically:

- The rows below the facility and the row containing the facility itself

$$\left(\frac{m+1}{2} - a\right) \times \left(\frac{|I|+1}{2} + a\right).$$

- The rows above the facility up to R_{mid}

$$\sum_{i=0}^{a-1}\left(\frac{|I|+1}{2} - a + 2i\right).$$

- The rows above R_{mid}

$$\left(\frac{m-1}{2}\right) \times \left(\frac{|I|+1}{2} - a\right).$$

Hence, B gains overall Voronoi region of:

$$\left(\frac{m+1}{2} - a\right) \times \left(\frac{|I|+1}{2} + a\right) + \sum_{i=0}^{a-1}\left(\frac{|I|+1}{2} - a + 2i\right) + \left(\frac{m-1}{2}\right) \times \left(\frac{|I|+1}{2} - a\right).$$

The second part of this equation can be simplified as follows:

$$\frac{|I|+1}{2} - a + 2\sum_{i=0}^{a-1} i.$$

And so, the final amount of the Voronoi region is equal to:

$$m\left(\frac{|I|+1}{2}\right) - a^2.$$

The amount of the Voronoi region for the same facility in R_{mid} is equal to

$$m\left(\frac{|I|+1}{2}\right).$$

Therefore, the lemma holds. □

Similar calculation for the case when the Voronoi region of a facility is in more than just one interval as illustrated in Figure 6, confirms the result of the previous lemmas. It is obvious now that for any $n \geq n_1$, moving a facility to another cell in the same interval decreases the Voronoi region for the facility (except for R_{mid}). Our goal is to compute a length for the grid,

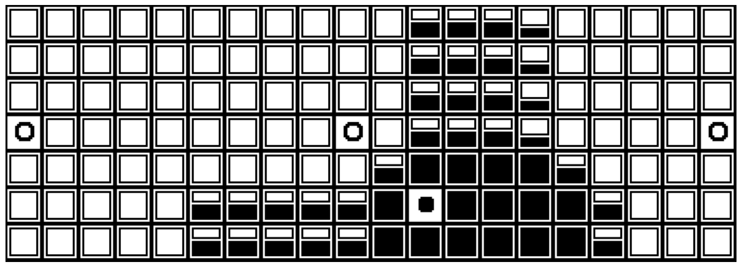


Figure 6: Voronoi region in more than one interval

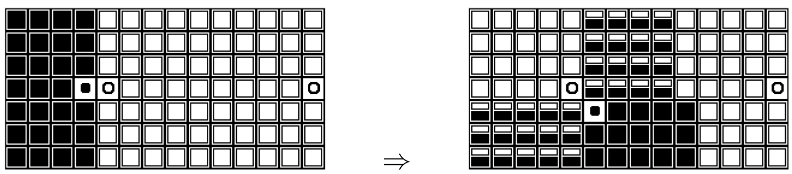


Figure 7: Transferring a facility from LHI

denoted by n_m, in which W wins the game with the winning margin of m in $G(m, n_m)$. Now the question is that if $n_1 = n_m$? Actually, there are some cases in which $n_m \leq n_1$ (for example $GVG_1(G(7, 29), 3)$). As a result and based on the number of cells which $\frac{1}{3}$ of them are owned by B, it is easy to show that n_m can be computed as follows:

$$n_m = \begin{cases} n_1 & ; \left(\frac{m+1}{2}\right) \bmod 3 = 0 \\ n_1 - (k-2) & ; \left(\frac{m+1}{2}\right) \bmod 3 = 1 \\ n_1 & ; \left(\frac{m+1}{2}\right) \bmod 3 = 2 \end{cases} \tag{14}$$

This equation along with the previous lemmas, decreases the number of possible facility movements to two cases which are called *valid movements*.

- Transferring a facility from LHI to its neighboring interval (IMIN or IMAX) including the column which contains W's facility.
- Transferring a facility from an IMIN interval to a neighboring IMAX one, including the column containing W's facility.

Definition 3. *The intersection of Voronoi regions of two facilities, is called the overlapping of these facilities.*

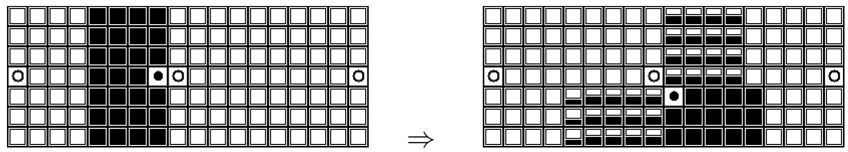


Figure 8: Transferring from IMIN to IMAX

The next observation summarizes the possible overlapping amounts of two facilities.

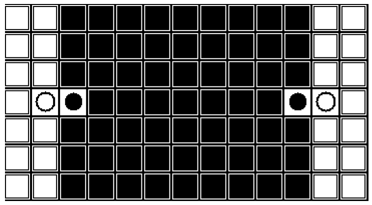
Observation 1. *Figure 9 presents all possible arrangements of two consecutive facilities when they have overlapping. The amount of overlapping for these possible cases in both odd and even intervals is computed.*

Lemma 3. *Suppose $G(m,n)$ is a grid in which, $n \geq n_m$. W wins $GVG_1(G,2)$ with the winning margin of m, if $2k \nmid n$. The game will end in a tie, when $2k \mid n$.*

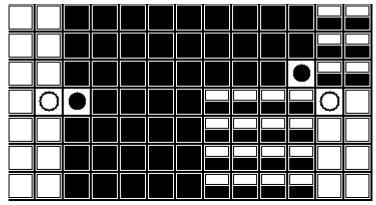
Proof. To prove this lemma, we will study different cases of Figure 9. In these cases, we denote the full intervals by I. To compute the Voronoi region of B in any of these cases, we first compute the Voronoi region of B's facilities one by one and independent from each other (assuming the absence of the other facilities of B). Then we compute the amount of overlapping for every two consecutive facilities of B (which are listed in Figure 9). Subtracting the sum of overlapping amounts form the sum of the Voronoi region of facilities and comparing it to the results of the simple strategy leads to the proof. As an example in the case of Figure 9(a), the Voronoi region of the left side facility of B is $m \times (\frac{|I|+1}{2})$. Furthermore, the Voronoi region of the right side facility of B independent from the left side facility is the same value. Hence the total value of the Voronoi region of B is $m \times |I|$. Considering the relation and the ratio between $|I|$, |LHI| and |RHI|, B gains values in Eq. (15) in case of playing according to Formula (7).

$$\begin{cases} m \times |I| & ; |\mathrm{RHI}| = |\mathrm{LHI}| = \frac{|I|-1}{2} \\ m \times |I| + \frac{m}{2} & ; |\mathrm{RHI}| = |\mathrm{LHI}| = \frac{|I|}{2} \\ m \times |I| + m & ; |\mathrm{RHI}| = |\mathrm{LHI}| = \frac{|I|+1}{2} \end{cases} \tag{15}$$

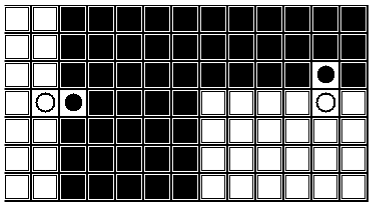
This implies that selecting the simple strategy by B, dominates the case illustrated in Figure 9(a). Similarly, in Figure 9(b) the Voronoi region for the left facility is



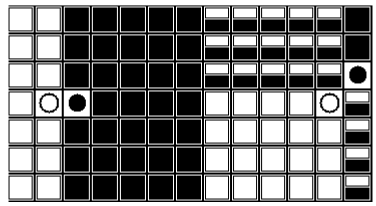
(a) Odd $:m$, Even $:m$



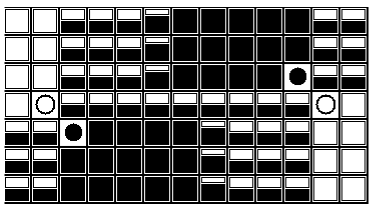
(b) Odd $:\frac{11m-7}{12}$, Even $:\frac{11m-7}{12}$



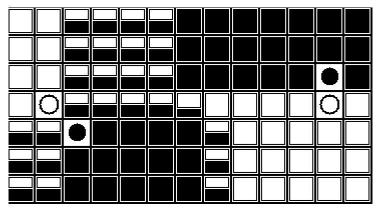
(c) Odd $:\frac{m-1}{2}$, Even $:\frac{m-1}{2}$



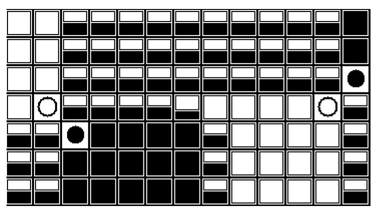
(d) Odd $:\frac{m-1}{6}$, Even $:\frac{m-1}{6}$



(e) Odd $:\frac{4m-3}{6}$, Even $:\frac{m-1}{2}$



(f) Odd $:\frac{m-1}{6}$, Even $:\frac{m-1}{6}$



(g) Odd $:\frac{m-1}{12}$, Even :0

Figure 9: Possible overlapping amounts of two facilities.

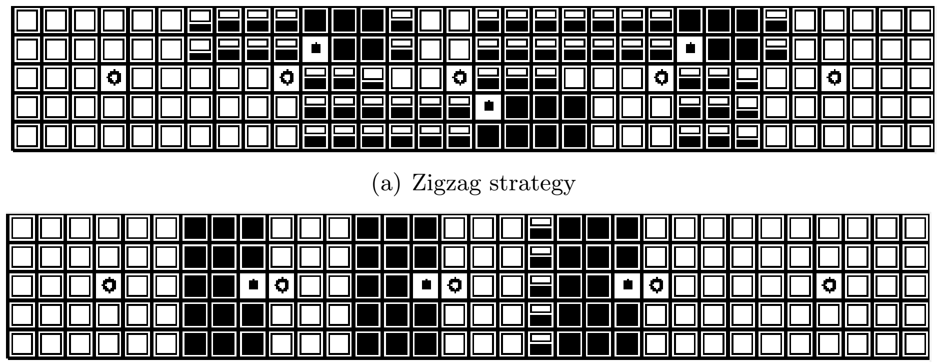
$$\left(\frac{3m-1}{8}\right) \times |I| + (m-1) \times \frac{|\mathrm{HI}|}{4} + \frac{3}{4}m - \frac{3}{4}$$

and for the right side facility is

$$m \times \left(\frac{|I|+1}{2}\right).$$

By considering $|I|$ to be odd, the total amount of B's Voronoi region is

$$m \times |I| - \frac{|I|}{4} + \frac{m}{3} - \frac{4}{3}.$$



(a) Zigzag strategy

(b) Simple strategy

Figure 10: Zigzag strategy vs. simple strategy

Clearly, this value is smaller than

$$m \times |I| + \frac{m}{2}.$$

The proof for the other cases are similar. $\square$

Lemma 4. *Let $G(m,n)$ be a grid in which $n \geq n_m$. B loses $GVG_1(G,3)$ with the minimum loss margin of m if $2k \nmid n$.*

Proof. The proof for this lemma is very similar to the proof of Lemma 3. The only difference is that in the game with $k = 3$ facilities, both of the valid movements are possible for B. Again, we should investigate all the possible cases of moving the facilities from R_{mid} and gain a larger Voronoi region. The calculations for these cases are similar to the one's in the lemma 3. These calculations for the possible cases will lead to the proof. $\square$

As mentioned, we started to move the facilities by one of the valid movements. Similar calculations indicate that when a movement starts with a valid one, it can only continue for at most three facility movement. Theorem 3 covers this problem.

Theorem 3. *For any odd m, any optional k and any $n \geq n_m$, W wins $GVG_1(G(m,n),k)$ with winning margin of m if $2k \nmid n$.*

Proof. It is clear that if B plays according to the simple strategy he loses the game by a loss margin of m. We are interested in the possibility of win or a smaller loss margin. To achieve these goals, consider the first two facilities of B. It is obvious that in order to achieve the efficiency of the **zigzag movement** of B's facilities, the Voronoi region for the first movement and the sum of Voronoi regions for the first and the second movements must dominate the Voronoi region of these facilities which their positions are selected based on the simple strategy. To demonstrate this formally, assume that the Voronoi region of the first movement of B is P' and the second one is Q'. Also suppose that by placing the same facilities in R_{mid} according to the simple strategy, B gains P and Q, respectively. It is clear that for a zigzag movement to be efficient, $|P'|+|Q'| > |P|+|Q|$ must holds. Considering this, for any k and m, in a grid with $n = n_m$ a zigzag movement must start with one of the valid movements and only grows if these conditions hold. So, to show the correctness of the proof, we will first start from the leftmost facility of B and will proceed to the right side of the grid one interval a step and will check the possibility of one or both of the valid movements. Now assume that the first valid movement is possible in the left half interval. If $k = 2$ or $k = 3$, by Lemma 3 and Lemma 4 respectively , we know that B loses the game with a loss margin of m. Similar reasonings for $k > 3$ indicates that moving at most three consecutive facilities from R_{mid} starting by the first valid movement, independent of the type of the neighboring intervals (IMIN or IMAX), is a non efficient action (Figure 4). that is, the amount of the Voronoi region for the very same facilities placed on R_{mid}, dominates the amount of the Voronoi region for the facilities on the zigzag. Likewise, the second valid movement will become non efficient in at most three moves. Hence, zigzag movement of just two facilities is not efficient (as for $k = 2$ in all cases). Similarly, three movements, in all possible cases is non efficient. As a result any zigzag movement of more than three facilities will be non efficient. Therefore, the proof is complete. □

Our Computational experiments have confirmed W will not lose the game for any $n_0 > mk - k + 1$. For $n < n_0$ the winner of the game will change from W to B and vice versa frequently. However, the winning margin of W for $n_0 \leq n < n_m$ is less that m in most cases. This problem along with the game on a grid with even width are currently open.

5 Conclusion and Future Works

In this paper, we have studied one round Voronoi game on one and two dimensional grids. As a result we proposed an optimal winning strategy for White (the first player) in both grids which guarantee the winning margin of m in the $G(m, n)$ where m, the width of the grid, is an odd number. Like other variations of the Voronoi game problem, several problems arises in this context. The most interesting problem is probably the game in the grid with the even width. Two dimensional k-round game which is a challenging problem in most contexts is an interesting open problem as well.

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